Unsteady Boundary Layer Flows of a Compressible Fluid (流体力学における非線型問題研究会報告集)

PAI, S.I.

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by

S. I. Pai

Research Professor, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland, U.S.A.

Visiting Professor, Institute of Space and Aeronautical Science, University of Tokyo, Tokyo, Japan

1. Introduction.

The unsteady boundary layer flows have not been studied thoroughly as the steady boundary layer problems because it is customary to use quasi-steady approach for unsteady boundary layer problem in practice. When the unsteady effects are large, such an quasi-steady approximation will not be good and we should study the unsteady boundary layer flow without the use of quasi-steady approach. In this paper, we first review some of the unsteady boundary layer analysis and then report some recent research work on unsteady boundary layer flow which have been carried out at University of Maryland. We shall consider only the starting of the boundary layer flow over a flat plate.

2. Brief history of unsteady boundary layer flow.

After Prandtl[1]* introduced the concept of boundary layer in 1904, Blasius worked out many exact solutions of the boundary layer equations. The first exact solution of unsteady boundary layer flow was given by Blasius in 1908 who considered the starting of a semi-infinite flat plate with a constant acceleration for an incompressible fluid.[2] Lord Rayleigh[3] obtained the exact

* This number refers to the number of reference in section 6.

solution of the impulsive motion of an infinite plate in 1911. Even though Rayleigh's solution was not based on boundary layer approximation, it contained the boundary layer solution if the Reynolds number is high. In 1944, Goertler[4] extended Blasius solution for finite acceleration as a power of time t. It is interesting to know that the impulsive motion and the finite acceleration motion represent two main classes of unsteady flow. Most of the exact solutions of unsteady boundary layer flows are obtained for finite acceleration cases which are the realistic cases. The impulsive motion for a semi-infinite plate has not been solved yet. In 1955, Stewartson[5] discussed the mathematical properties of the impulsive motion of a semi-infinite plate and found two essential singularities: one at $\xi=0$ and the other at $\xi=x/Ut=1$. Between $\xi=0$ and $\xi=1$, the flow depends both on x, the direction along the plate and y, the direction perpendecular to the plate. For $\xi>1$, the flow is independent of x and is Rayleigh's solution. Stewartson did not find the solution for $\xi<1$. Lam and Crocco[6] used the iteration method to find the solution of impulsive motion of a semi-infinite plate for $\xi<1$ and found that the solution diverges after 18 iteration.

The first unsteady boundary layer solution of a compressible fluid was given by Illingworth[7] in 1950. He considered an infinite plate and introduced the well known approximation of $\rho\mu$ = const. where ρ is the density of the fluid and μ is its coefficient of viscosity. Moore[8] was the first one to study the unsteady boundary layer over a semi-infinite plate of arbitrary free stream velocity. He used the approximation $\rho\mu$= const. and decoupled the equation of motion from the energy equation.

He was able to show that the quasi-steady approach is a first approximation of the exact solution. He considered only the cases of constant wall temperature or insulated wall.

Recently we look[9] into the data of $\rho\mu$ and $\rho\mu/Pr$ for the equilibrium dissociating air given by Cohen[10] and find that they are linear with respect to enthalpy h, i.e.,

$$\rho\mu = a_1 - b_1 h \tag{1}$$

$$(\rho\mu/Pr) = a_2 - b_2 h \tag{2}$$

where Pr is the Prandtl number and a_1, a_2, b_1 and b_2 are constants. For equilibrium dissociating air of the temperature range of $80^\circ F$ to $880^\circ F$ and a pressure range from 0.01 to 100 atmospheres, the following values of these coefficients are found:

$$\begin{aligned} a_1 &= 20.00 \times 10^{-5}\ \text{slug}^2/\text{sec-ft}^4;\quad a_2 = 27.5 \times 10^{-5}\ \text{slug}^2/\text{sec-ft}^4 \\ b_1 &= 1.07 \times 10^{-11}\ \text{slug}^2\text{-sec/ft}^6;\quad b_2 = 1.30 \times 10^{-11}\ \text{slug}^2\text{-sec/ft}^6 \end{aligned} \tag{3}$$

where the units of enthalpy h is in ft^2/sec^2.

We solve the unsteady boundary layer equations with these realistic values of transport coefficients given in Eqs.(1) to (3) and compare them with the simplified results of $\rho\mu$ = const.

3. Fundamental equations and initial and boundary conditions.

The fundamental equations of unsteady boundary layer flow in coordinates fixed to the plates are as follows:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{4}$$

$$\rho\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = \rho\left(\frac{\partial \bar{h}}{\partial t} + \bar{u}\frac{\partial \bar{h}}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{5}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{6}$$

$$p = p(x,t) = R\rho T \tag{7}$$

where bar refers to the value of free stream. The other symbols are standard notation with x in the direction of the plate. The boundary and initial conditions associated with Eqs. (4) to (7) are as follows:

$$\begin{aligned}
&\text{(i)} && \text{at } y=0: u=v=0,\ h=h_0(x,0,t)=\text{given function of } x \text{ and } t\\
&\text{(ii)} && \text{at } y=\infty: u=\bar{u}(x,t),\ h=h_\infty=\bar{h}(x,t)\\
&\text{(iii)} && \text{at } t=0: u=v=0,\ h(x,0,0)=h_0(x,0,0),\\
& && \qquad h(x,y,0)=\bar{h}(x,0) \text{ where } y\neq 0.\\
&\text{(iv)} && \text{at } x=0: u=\bar{u}(0,t),\ h=\bar{h}(0,t).
\end{aligned} \tag{8}$$

4. Transformations and method of solution.

We are going to solve Eqs. (4) to (7) with the initial and boundary conditions (8). The equations may be simplified by using the new varaibles Y, X, t' instead of y, x, t such that

$$Y = \int_0^y \left(\frac{\rho}{\rho_\infty}\right) dy; \quad X = x; \quad t' = t \tag{9}$$

The final solutions of the unsteady boundary layer equations depend on the form of the free stream velocity $\bar{u}$ and free stream enthalpy $\bar{h}$. For arbitrary function of $\bar{u}$ and $\bar{h}$, the amount of numerical computation would be so large that the best high speed computing machine available would be insufficient. Hence we shall consider the case of constant acceleration with $\bar{u}=At'$, $\bar{h}$=const.

but the wall enthalpy may be an arbitrary function of X and t'. Under these conditions, all the unknowns will be functions of two independent variables ξ and η defined as follows:

$$\xi = \frac{X}{At'^2}, \qquad \eta = Y(\nu_0 t')^{-1/2} \tag{10}$$

The dependent variables may be expressed in terms of a stream function ψ and enthalpy h and we shall use the expansions:

$$\psi = (\nu_0 t')^{1/2} A \sum_{n=1}^{\infty} t'^n g_n(\xi,\eta) \tag{11}$$

$$h = \sum_{n=0}^{\infty} t'^n j_n(\xi,\eta) \tag{12}$$

Furthermore, we shall limit to the case of small time t', i.e., large ξ. The functions g_n and j_n may be written as follows:

$$g_n = \sum_{m=0}^{\infty} (\tfrac{1}{\xi})^m g_{nm}(\eta) \tag{13}$$

$$j_n = \sum_{m=0}^{\infty} (\tfrac{1}{\xi})^m j_{nm}(\eta) \tag{14}$$

Numerical solutions of g_{nm} and j_{nm} for various initial and boundary conditions have been worked out for various cases in reference 9.

5. Results.

The detailed results of our computation are given in reference 9. Here we discuss two cases which will show the effects of the variation of $\rho\mu$ and the accuracy of quasi-steady approximation in general.

(i) Insulated semi-infinite flat plate whose initial temperature equals to the free stream temperature.

The first order velocity terms $g'_{10}(\eta)$ is the Blasius solution, i.e., the quasi-steady approximation. But the effect of $\rho\mu \neq$ constant shows in the higher order terms g'_{30} etc. which is in general not negligible. There is considerable difference in temperature profiles for the case with $\rho\mu \neq$ const. from that with $\rho\mu$ = constant. Fig.1 shows that the wall temperature for our more accurate analysis is much lower than that in case of assuming Reynolds analogy.

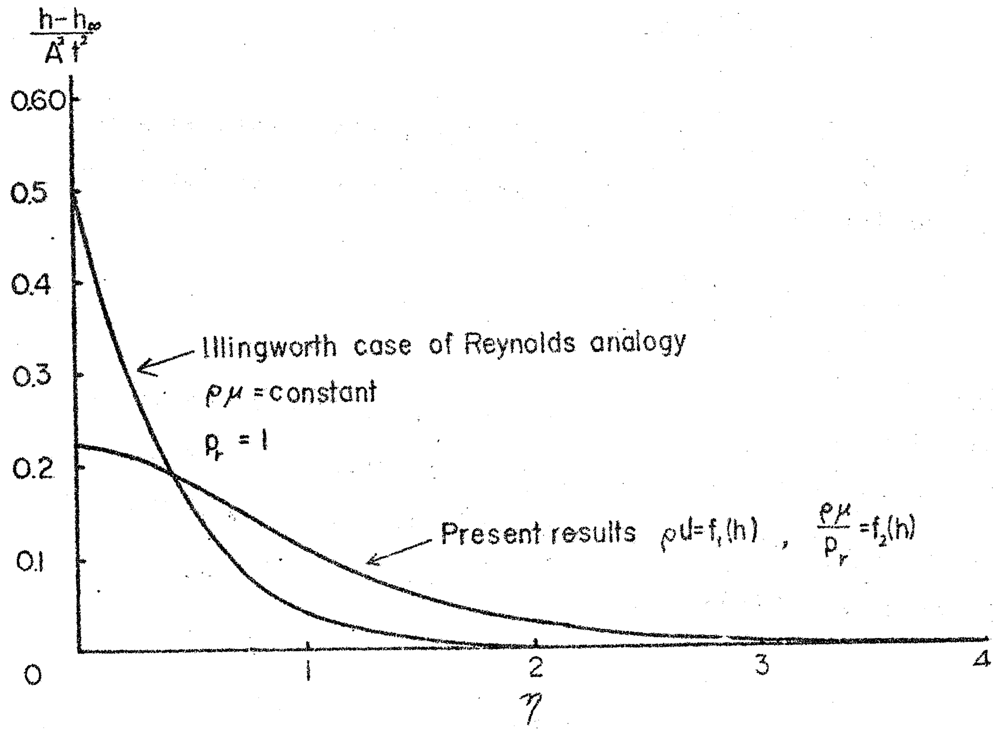

Fig. 1 Comparison of enthalpy distributions with constant and variable values of $\rho\mu$ and $\rho\mu/P_r$ (Insulated plate)

(ii) The temperature of the plate is a linear function of time t':

$$h_0 = 3.16 \times 10^6 + 3.00 \times 10^6 t', \quad h_\infty = 3.16 \times 10^6$$

where the units of h is ft^2/sec^2.

In the present case, we found that the quasi-steady approximation is reasonably good for velocity profile and shearing stress but it is definitely not good for temperature profiles and heat transfer rate. We compare one case of constant temperature with variable wall temperature at an instant when their wall temperatures are equal and find that the difference of heat transfer rate is over 10 times.

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